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# *Curves on Quintic Scrolls.*

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## INTRODUCTION.

The chief aim of this paper is to prove, for scrolls of order 5, Professor Story's formula for the number of intersections of any two curves lying on the same ruled surface, which may be stated as follows: If a curve of order $a$ lies entirely on a ruled surface of order $\mu$, say $R_\mu$, and meets each generator of $R_\mu$ in the same number of points, say $\alpha$, this curve will be designated by the symbol $C_{a,\alpha}$ and $\alpha$ may be called the *rank* of the curve; then the number of intersections[*] of $C_{a,\alpha}$ with any other curve on $R_\mu$, say $C_{b,\beta}$, of order $b$ and rank $\beta$, is given by the formula

$$[C_{a,\alpha}\,C_{b,\beta}] = a\beta + b\alpha - \mu\alpha\beta.$$

In 1883[†] Professor Story proved this formula for $\mu = 2$ and $\mu = 3$. In 1900 I proved it for $\mu = 4$ and gave, in that paper,[‡] Professor Story's three theorems which must be proved before the formula can be established, viz.:

THEOREM I. If $\alpha$ be the number of points of $C_a$ on an arbitrary generator, then there are $\alpha$ points of $C_a$ on each generator.

THEOREM II. If $C_{a,\alpha}$ is the complete intersection of $R_\mu$ and $S_\nu$ ($S_\nu$ being any surface of order $\nu$) and $C_{b,\beta}$ is any curve on $R_\mu$ that has no component in common with $C_{a,\alpha}$, then $[C_{a,\alpha}\,C_{b,\beta}] = a\beta + b\alpha - \mu\alpha\beta$.

THEOREM III. If $C_{a,\alpha}$ is irreducible and the partial intersection of $R_\mu$ and $S_\nu$, $C_{a',\alpha'}$ being the residual intersection, and if the formula holds for each irreducible component of $C_{a',\alpha'}$ with an arbitrary curve $C_{b,\beta}$ on $R_\mu$, it also holds for $C_{a,\alpha}$ with $C_{b,\beta}$.

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[*] It is to be understood here that if two curves pass through the same point on a multiple line or multiple curve they are *not* to be regarded as intersecting there unless they lie on the same sheet.

[†] On the Number of Intersections of Curves Traced on a Scroll of Any Order. *Johns Hopkins University Circulars*, August, 1883.

[‡] Geometry on Ruled Quartic Surfaces. *Proc. Amer. Acad. of Arts and Sciences*, Vol. XXXVI, No. 3.

The proofs of Theorems II and III are given in the paper cited, and Dr. Story has recently succeeded in proving Theorem I (called the fundamental theorem) for all ruled surfaces, and in also proving his formula for ruled surfaces of any order, $\mu$, that have a multiple linear director of order $\mu - 1$ or $\mu - 2$, and these cases include seven of the fifteen cases of quintic scrolls enumerated by Schwarz.* These cases, however, as well as the proof of Theorem I in many cases, will be given here since these proofs are, in general, somewhat different.

The formula holds for each generator (which is a $C_{1,0}$ since it does not meet an arbitrary generator) with any curve $C_{a,\alpha}$, for $[C_{a,\alpha}\, C_{1,0}] = 0 + \alpha - 0 = \alpha$, and since every plane curve is either the complete intersection of the scroll and the plane or the partial intersection by a plane that cuts out otherwise one or more generators, by Theorems II and III the formula holds for all plane curves on an irreducible scroll of any order. It is not possible for a plane to cut an irreducible scroll in two distinct proper curves and, as Schwarz has pointed out, no scroll of order higher than four can have more than one simple conic, since two conics not in the same plane which correspond, point for point, determine a scroll of the fourth order.

The necessary multiple curve, or aggregate of multiple curves and rectilinear directors, on $R_\mu$ (which determine the scroll) will be called the frame and designated by the symbol $F(C^m_{a,\alpha},\ C^n_{b,\beta},\ \ldots.)$, where $m$, $n$, .... denote the multiplicities of the curves or linear directors and are omitted if the curves or linear directors are simple. The order of the curve must be multiplied by the multiplicity in reckoning the number of intersections, *i. e.*,

$$[C^m_{a,\alpha}\, C^n_{b,\beta}] \equiv ma\beta + nb\alpha - \mu\alpha\beta.$$

It will be well to remember that this formulà may not give all the actual intersection of two curves in space; if $\mu$ be chosen large enough so that the two curves lie on the same scroll, it may happen that both curves pass through the same point on a multiple curve but lie on different sheets there, and this point, which is a point of actual intersection in space, is not one of the points of intersection on the scroll as given by the formula, in accordance with the foot-note on page 421. It may, in like manner, happen that the number of intersections of any two curves considered as lying on one scroll, may be different from the number of their intersections when regarded as lying on another scroll.

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* Über die geradlinigen Flächen fünften Grades. *J. für Math. von Crelle*, B. 67.

If $S_\nu$ is a surface of order $\nu$ used to cut, from the scroll, any desired curve, the number of points that determine $S_\nu$ is $\frac{1}{6}(\nu+1)(\nu+2)(\nu+3)-1$; but if $\nu \geqq 5$ we must be sure that $S_\nu$ does not break up in the given scroll, $R_5$, and a surface $S_{\nu-5}$, and to do this we take $\frac{1}{6}(\nu-4)(\nu-3)(\nu-2)$ points of $S_\nu$ off the scroll $R_5$, *i. e.*, one more than enough to determine the surface $S_{\nu-5}$; as this number vanishes for $\nu = 2$, 3, or 4, we have for $\nu \geqq 2$

$$[\tfrac{1}{6}(\nu+1)(\nu+2)(\nu+3)-1]-[\tfrac{1}{6}(\nu-4)(\nu-3)(\nu-2)] = \tfrac{5}{2}\nu(\nu-1)+4$$

arbitrary points available.

For simplicity the scrolls of order 5 will be considered in the order given by Schwarz.

### (A) SCROLLS OF DEFICIENCY ZERO. ($\rho = 0$.)

### I. *Quintic Scroll with a Quadruple Linear Director.* $F(C^4_{1,1})$.

1. Only one scroll with a quadruple line is given by Schwarz, but there are two distinct species, viz., $(I_1)$ without a simple director and $(I_2)$ with a simple director, the equations in homogeneous coordinates $x$, $y$, $z$ and $s$ being of the form

$$U_4 z + V_4 s + W_5 = 0 \qquad \text{for } (I_1)$$

and

$$U_4 z + V_4 s \qquad = 0 \qquad \text{for } (I_2),$$

where $U$, $V$, and $W$ are homogeneous polynomials in $x$ and $y$ of degrees indicated by the subscripts, the quadruple line being $x = 0$, $y = 0$ and the simple director in $(I_2)$ being $z = 0$, $s = 0$.

2. *Proof of Theorem* I. In either case, a plane through the quadruple line, $C^4_{1,1}$, cuts out otherwise one generator from the scroll and any curve, $C_b$, on the scroll is met by this plane in a certain number of points, say $\beta$, on the generator and in $b-\beta$ points on $C^4_{1,1}$; as the plane is revolved about $C^4_{1,1}$ it cuts out, in succession, each generator of the scroll, and since there are always $b-\beta$ points of $C_b$ on $C^4_{1,1}$ there are $\beta$ points of $C_b$ on each generator.

3. The formula holds for the frame, $C^4_{1,1}$, since $[C^4_{1,1}\, C_{b,\beta}] = 4\beta + b - 5\beta = b - \beta$, and we have just seen that $C_{b,\beta}$ has $b-\beta$ points on $C^4_{1,1}$.

Now if a cone of order $b$, with its vertex at an arbitrary point on $C^4_{1,1}$, be passed through any curve $C_{b,\beta}$, the complete intersection of the scroll and the cone will be $C_{b,\beta}$, the four generators through the vertex, each of which is a $\beta$-tuple edge of the cone, and $C^4_{1,1}$, which is a $(b-\beta)$-tuple edge of the cone and counts for $4(b-\beta)$ lines in the intersection; for this gives $4(b-\beta)+4\beta+b=5b$, the order of complete intersection. Since the formula holds for every generator and for $C^4_{1,1}$, by Theorem III it holds for every curve $C_{b,\beta}$, plane or twisted.

4. *Plane Curves.* On the species $(I_2)$ neither a plane cubic nor a conic can exist, since a plane through two generators must contain also the simple director and consequently also the other two generators through the point where the plane cuts the quadruple line; in other words, a plane through the simple director cuts out, otherwise, the four generators, one on each sheet, that pass through the point where the plane cuts the quadruple line. One and only one conic may exist on species $(I_1)$ if, at some point of the quadruple line, three generators lie in the same plane; the plane of this conic will therefore be a triple tangent plane to the scroll and the conic will pass through the three points of tangency—these being the points where the conic meets the three generators in distinct points, and the other point of common intersection being on the quadruple line at which the conic lies on the sheet not having a generator in this plane. A plane through two generators on $(I_1)$ will cut out a plane cubic $C_{3,1}$ having a double point where the plane cuts the quadruple line, and no other kind of a plane cubic exists on this scroll.

The only kind of a plane quartic that exists on either species is a $C_{4,1}$ having a triple point on $C^4_{1,1}$, whose plane cuts out one generator, and the only kind of a plane quintic that exists is a $C_{5,1}$ having a quadruple point on $C^4_{1,1}$, whose plane does not contain any line on the scroll. The line of intersection of the plane of a quartic and the plane of a quintic pierces the scroll five times; once where the generator in the plane of the quartic meets the quintic, and four times where the curves intersect, $[C_{5,1}\ C_{4,1}] = 5 + 4 - 5 = 4$.

On species $(I_1)$ a plane quartic and plane cubic intersect in $[C_{4,1}\ C_{3,1}] = 4 + 3 - 5 = 2$ points; the other three points where the line common to their planes pierces the scroll being the one where the generator in the plane of the quartic meets the cubic and the two where the two generators in the plane of the cubic meet the quartic. If a conic exists on the scroll $(I_1)$, it meets every plane quintic in two points, every quartic in one point, and does not meet a cubic at all, $[C_{2,1}\ C_{3,1}] = 2 + 3 - 5 = 0$.

5. *Twisted Curves.* Every curve on the scroll must meet the quadruple line; for we have seen that every curve, $C_{b,\beta}$, meets the quadruple line in $b-\beta$

points and $b-\beta \geqq 1$ since for species $(I_1)$ two generators lie in a plane and $\beta \leqq \frac{b}{2}$, and for species $(I_2)$ four generators lie in a plane and $\beta \leqq \frac{b}{4}$. Moreover, in the equation of species $(I_1)$ there are only $5+5+6=16$ terms or 15 arbitrary parameters, and in species $(I_2)$ only nine arbitrary parameters, and in order to make a quintic scroll contain even the cubic we must have $3\times 5+1=16$ arbitrary parameters, unless the quadruple line bears some relation to the curve, *i. e.*, an arbitrary line can not be chosen to be the quadruple line of the scroll and the scroll be made to pass through an arbitrary curve, even of the lowest order.

On species $(I_2)$, no cubic can exist, since there are four generators in a plane and each generator must meet every curve at least once, but there can be a $C_{4,1}$ meeting the quadruple line, $C^4_{1,1}$, in three points, a $C_{5,1}$ meeting $C^4_{1,1}$ four times, etc., up to curves of order 8, of which we may have a $C_{8,1}$ meeting $C^4_{1,1}$ seven times, or a $C_{8,2}$ meeting $C^4_{1,1}$ six times; so that the curves on species $(I_2)$ are arranged in groups of four, according to their ranks.

On species $(I_1)$ every cubic is a $C_{3,1}$, and if there is no conic on the scroll only two generators lie in a plane, so that there may be a $C_{4,2}$ as well as a $C_{4,1}$, whereas if the conic exists there are three generators in one plane and no $C_{4,2}$ can exist. Accordingly, on this species $(I_1)$ $\beta \leqq \frac{b}{3}$ if the conic exists and $\beta \leqq \frac{b}{2}$ if the conic does not exist (*i. e.*, if the scroll does or does not have a triple tangent plane, respectively).

The number of intersections of any curve $C_{b,\beta}$ with a $C_{5,1}$ is equal to the order of the curve, *i. e.* $[C_{b,\beta}\, C_{5,1}] = b+5\beta-5\beta = b$. The number of intersections of any two curves $C_{a,\alpha}$ and $C_{b,\beta}$, on either species $(I_1)$ or $(I_2)$, is equal to the order of the residual curve of intersection of the two cones, of orders $a$ and $b$ respectively, through these two curves from an arbitrary point on the quadruple line as a vertex: for, the cones intersect in a curve of order $ab$; each of the four generators through the vertex counts as an $\alpha$-tuple edge on one cone and a $\beta$-tuple edge on the other, while the quadruple line counts for an $(a-\alpha)$-tuple edge on one and a $(b-\beta)$-tuple edge on the other, so that the residual curve is of order

$$ab-[4\alpha\beta+(a-\alpha)(b-\beta)] = a\beta+b\alpha-5\alpha\beta = [C_{a,\alpha}\, C_{b,\beta}].$$

This theorem is true for a scroll of any order, $R_\mu$, having a director line of multiplicity $(\mu - 1)$, for there are $\mu - 1$ generators through the common vertex of the cones and the residual curve is of order

$$ab - [(\mu - 1)\alpha\beta + (a - \alpha)(b - \beta)] = a\beta + b\alpha - \mu\alpha\beta,$$

which is $[C_{a,\alpha}\, C_{b,\beta}]$.

## II. *Quintic Scroll with a Double Sextic Curve having a Triple Point.* $F(C^2_{6,3})$.

Every generator meets the double sextic three times; for, an arbitrary plane through one generator cuts out otherwise a plane quartic which is met four times by the generator, once at the point of tangency of the plane and three times on the double curve. The double sextic is therefore a $C^2_{6,3}$.

1. *Proof of Theorem* I. Pass a cubic surface, $S_3$, through the double sextic $C^2_{6,3}$. This we can do by taking the triple point and 16 other points of $C^2_{6,3}$ as 17 of the 19 points necessary to determine $S_3$; $C^2_{6,3}$ will then meet $S_3$ three times at the triple point and therefore 19 times in all, and consequently will lie entirely on $S_3$. As we have two more points for the determination of $S_3$ that can be arbitrarily chosen, a double infinity of cubic surfaces can be passed through $C^2_{6,3}$. Now, every generator meets $S_3$ three times on $C^2_{6,3}$, so that if we choose our two points, one on each of two generators, these generators must lie entirely on $S_3$ and the residual intersection of $S_3$ and the scroll will be $15 - (12 + 2) = 1$, *i. e.*, another generator; moreover, it is easy to see that any point of the intersection of $S_3$ and the scroll that is not a point of $C^2_{6,3}$, lies on some generator of the scroll, which generator therefore meets $S_3$ four times, *i. e.*, lies entirely on $S_3$, so that the residual intersection of $S_3$ that passes through $C^2_{6,3}$ consists always of three generators, two of which may be arbitrarily chosen. Consider any curve, $C_b$, on the scroll; it meets $S_3$ in $3b$ points, of which a fixed number, say $k$, lie on $C^2_{6,3}$ and the remaining $3b - k$ lie on the three generators; if $p$ is the number of points of $C_b$ on any generator, it is evident that $0 \overset{=}{<} p \overset{=}{<} b - 1$ and $p$ can have, at most, $b$ different values. For convenience we shall say that all generators for which $p$ has the same value belong to the same set, a finite set or an infinite set, according as the number of generators in it is finite or infinite. There are, at most, $b$ different sets and, since $b$ is finite, at least one set must be infinite. Since there are only a finite number of generators in all the finite sets taken together, we can always choose $S_3$ so that no generator of any finite set will form part of the intersection. Suppose there are two infinite sets with values of $p$ equal to $\beta$ and $\gamma$ respectively; having two

points at our disposal we can make $S_3$ cut out two $\beta$-generators in such a way as to avoid a finite set, and the third generator cut out will be either a $\beta$-generator or a $\gamma$-generator; if it is a $\beta$-generator then $3\beta = 3b - k$, and if it is a $\gamma$-generator $2\beta + \gamma = 3b - k$. In a similar manner we can make an $S_3$ cut out two $\gamma$-generators, and we will have either $3\gamma = 3b - k$ or $2\gamma + \beta = 3b - k$ according as the third generator is a $\gamma$- or a $\beta$-generator. Either of the first two conditions combined with either of the second two gives $\gamma = \beta$, so that there is only one infinite set. If we assume that there are three infinite sets, $\beta$, $\gamma$, and $\delta$, and put for convenience $3b - k \equiv h$, the only arrangement differing from the above is:

$$\left\{\begin{array}{l} 2\beta + \gamma = h \\ 2\gamma + \delta = h \\ 2\delta + \beta = h \end{array}\right\} \text{ or } \left\{\begin{array}{l} \gamma = h - 2\beta \\ \delta = h - 2\gamma = -h + 4\beta \\ \beta = h - 2\delta = \quad 3h - 8\beta \\ h = 3\beta \end{array}\right\} \text{ and therefore } \left\{\begin{array}{l} \gamma = \beta \\ \delta = \beta \\ h = 3\beta = 3b - k \\ k = 3b - 3\beta \end{array}\right\},$$

which shows that only one infinite set exists. As this method can be extended to any finite number of sets, and $b$ is finite, it is certain that only one infinite set exists. If now one point be chosen on a generator of a finite set, say an $\alpha$-generator, and the other point so chosen on a $\beta$-generator of the infinite set that the third generator is also a $\beta$-generator, then

$$2\beta + \alpha = 3b - k = 3\beta, \text{ or } \alpha = \beta.$$

Therefore there is only one set of any kind on the scroll and every generator meets $C_b$ in $\beta$ points, while $C^2_{6,3}$ meets $C_{b,\beta}$ in $3b - 3\beta$ points. Since

$$[C^2_{6,3}\ C_{b,\beta}] = 12\beta + 3b - 15\beta = 3b - 3\beta,$$

the formula holds for the frame.

2. *Plane Curves.* Every plane quintic, $C_{5,1}$, has six double points, where its plane cuts the double sextic, and intersects the double sextic, therefore, in twelve points, twice at each double point, $[C_{5,1}\ C^2_{6,3}] = 15 + 12 - 15 = 12$. Every plane quartic is a $C_{4,1}$, having three double points, and meets $C^2_{6,3}$ in nine points, twice at each double point and once at each of the three points where the generator in the plane of the quartic meets the $C^2_{6,3}$, and $[C^2_{6,3}\ C_{4,1}] = 12 + 12 - 15 = 9$.

A plane through two generators cuts out a plane cubic, $C_{3,1}$, with a double point, that meets $C^2_{6,3}$ in six points, twice at the double point and once each at two points on each generator, $[C^2_{6,3}\ C_{3,1}] = 12 + 9 - 15 = 6$. The plane of the cubic is a double tangent plane. If a triple tangent plane exists, it cuts out three generators and a conic, and therefore only one such plane can exist; the conic will meet $C^2_{6,3}$ in three points, one on each generator, and the other point of intersection of the conic with each generator is a point of tangency.

$$[C^2_{6,3}\ C_{2,1}] = 12 + 6 - 15 = 3.$$

3. *Twisted Curves.* Let $S_\nu$ be the surface that cuts out any curve $C_{b,\beta}$ from the scroll, and take $\nu$ large enough so that $S_\nu$ can be made to cut out $C^2_{6,3}$ and at the same time have a residual intersection, outside of generators, of order less than $b$, *i. e.*, $r \leqq b - 1$. By taking the triple point of $C^2_{6,3}$ as one point of $S_\nu$, it will be necessary to take only $6\nu - 2$ other points of $S_\nu$ on $C^2_{6,3}$, or $6\nu - 1$ in all; $b\nu + 1$ points of $C_{b,\beta}$ must lie on $S_\nu$, but as $3(b - \beta)$ points of $C_{b,\beta}$ lie on $C^2_{6,3}$ only $b\nu + 1 - 3(b - \beta)$ additional points of $S_\nu$ need be taken on $C_{b,\beta}$. The number of arbitrary points of $S_\nu$ remaining is

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{(b + 6)\nu - 3(b - \beta)\}.$$

At least two generators lie in one plane, and $\beta \leqq \dfrac{b}{2}$, so that every cubic is a $C_{3,1}$; if we take $\nu = 4$, we can make $S_4$ cut out $C_{3,1}$ and $C^2_{6,3}$ and have four arbitrary points left; the residual will consist entirely of generators, since each generator already meets $S_4$ three times on $C^2_{6,3}$ and once on $C_{3,1}$. Since the formula holds for $C^2_{6,3}$ and the generators it holds for $C_{3,1}$.

In general, if we choose $\nu = \dfrac{2b + 11 + n}{5}$, where $n = 1, 2, 3, 4$ or $5$ according as $2b + 11$ is congruent to 4, 3, 2, 1 or 0 (mod. 5), respectively, and make $S_\nu$ cut out $n$ generators in each case, we will have for the order of the residual,

$$r = 2b + 11 + n - (b + 12 + n) = b - 1.$$

To do this we must have

$$\frac{5}{2}\left(\frac{2b + 11 + n}{5}\right)\left(\frac{2b + 6 + n}{5}\right) + 4 - (b + 6)\left(\frac{2b + 11 + n}{5}\right) + 3b - 3\beta - n\left(\frac{2b + 11 + n}{5} - 2 - \beta\right) \geqq 0,$$

*i. e.*,

$$2(9 - n)b + (3 - n)n - 10(3 - n)\beta - 26 \geqq 0.$$

Since $\beta \leqq \frac{b}{2}$, this condition is satisfied, in the five different cases, by the following values of $b$, viz.:

1st Case: $n = 1$;

$$\nu = \frac{2b + 12}{5};\ b = 4,\ 9,\ 14,\ \ldots\ \left.\vphantom{\frac{2b}{5}}\right\} 16b - 20\beta - 24 \geqq 0, \text{ satisfied for } b \geqq 4.$$

2nd Case: $n = 2$;

$$\nu = \frac{2b + 13}{5};\ b = 6,\ 11,\ 16,\ \ldots\ \left.\begin{matrix} 14b - 10\beta - 24 \geqq 0 \\ 10\beta \leqq 5b;\ 9b \geqq 24 \end{matrix}\right\}, \text{ satisfied for } b \geqq 3.$$

3rd Case: $n = 3$;

$$\nu = \frac{2b + 14}{5};\ b = 3,\ 8,\ 13,\ \ldots\ \left.\vphantom{\frac{2b}{5}}\right\} 12b - 26 \geqq 0, \text{ satisfied for } b \geqq 3.$$

4th Case: $n = 4$;

$$\nu = \frac{2b + 15}{5};\ b = 5,\ 10,\ 15,\ \ldots\ \left.\begin{matrix} 10b + 10\beta - 30 \geqq 0 \\ \beta \geqq 1;\ 10b \geqq 20 \end{matrix}\right\}, \text{ satisfied for } b \geqq 2.$$

5th Case: $n = 5$;

$$\nu = \frac{2b + 16}{5};\ b = 7,\ 12,\ 17,\ \ldots\ \left.\begin{matrix} 8b + 20\beta - 36 \geqq 0 \\ \beta \geqq 1;\ 8b \geqq 16 \end{matrix}\right\}, \text{ satisfied for } b \geqq 2.$$

Therefore, for all twisted curves on the scroll, we can choose $\nu$ so that $r \leqq b - 1$, and since we have proved the formula for $b = 3$, it holds for all values of $b$.

## III. *Quintic Scroll with a Triple Linear Director and a Double Twisted Cubic.* $F(C^{3}_{1,1}\ C^{2}_{3,1})$.

A plane through the two generators that meet at an arbitrary point on the double cubic, cuts out the triple line, and cuts the cubic in two other points which, being double points of the section, must lie on the triple line; *i. e.*, the triple line meets the double cubic twice, and each generator meets the double cubic once, so that the double cubic is a $C^{2}_{3,1}$ and the triple line is a $C^{3}_{1,1}$.

1. *Proof of Theorem* I. Pass a quadric surface, $S_2$, through the double cubic, $C^{2}_{3,1}$, by choosing seven points of $S_2$ on $C^{2}_{3,1}$. Since two points of the triple line, $C^{3}_{1,1}$, lie on $C^{2}_{3,1}$, $C^{3}_{1,1}$ will lie entirely on $S_2$, if one more point of $S_2$ be chosen on $C^{3}_{1,1}$. The total intersection of $S_2$ and the scroll will then consist of $C^{2}_{3,1}$, $C^{3}_{1,1}$ and one generator $(6 + 3 + 1 = 10)$ and this generator can be selected at will by choosing the ninth point for the determination of $S_2$ on it. Now any curve, $C_b$, on the scroll meets $C^{2}_{3,1}$ and $C^{3}_{1,1}$ together in a fixed number of points, say $k$, and therefore must meet each generator in the same number of points, say $\beta$, where

$$\beta = 2b - k \quad \text{or} \quad k = 2b - \beta.$$

2. *Plane Curves.* Every plane quintic is a $C_{5,1}$ having one triple point where its plane cuts the triple line and three double points where its plane cuts the double cubic; if the plane goes through one of the two points common to $C^3_{1,1}$ and $C^2_{3,1}$, the triple point of the quintic is modified by having two branches tangent, making the point equivalent to four double points. Every plane through one and only one generator cuts out a plane quartic, a $C_{4,1}$, having three double points, one on the triple line where the generator in its plane passes through and two on the double cubic where its plane cuts the double cubic not on the generator. Regarded as lying in the plane the generator meets the quartic four times, twice at the double point on the triple line, once where the generator meets the double cubic, and once at the point of tangency of the plane and scroll; but regarded as lying on the scroll, the generator meets the quartic only once, at the point of tangency of the plane and scroll, since this is the only one of the four where the generator and quartic lie on the same sheet, $[C_{4,1}\, C_{1,0}] = 0 + 1 - 0 = 1$. Every plane cubic is a $C_{3,1}$ having a double point where its plane cuts the double cubic, not on either of the two generators in its plane. There is one and only one true conic on this scroll, if there exists a triple tangent plane containing all three generators at some point on the triple line, in which case the conic meets each generator once on the double cubic and once at a point of tangency of plane and scroll.

3. *Twisted Curves.* Pass a cubic surface, $S_3$, through $C^2_{3,1}$ and $C^3_{1,1}$, by choosing 10 points $S_3$ on $C^2_{3,1}$ and two additional points of $S_3$ on $C^3_{1,1}$. Now $\beta \leqq \frac{b}{2}$ so that every cubic curve is a $C_{3,1}$ and has five points on $C^2_{3,1}$ and $C^3_{1,1}$ together; therefore by choosing five additional points of $S_3$ on $C_{3,1}$, $S_3$ will cut out $C_{3,1}$, the complete intersection being $C^2_{3,1}$, $C^3_{1,1}$, $C_{3,1}$ and three generators since every generator will already meet $S_3$ three times on the triple line and two curves together, and no other point of the residual can exist unless the generator through that point lies entirely on $S_3$. The formula holds for the frame, since the triple line lies on both sheets through the double cubic at each of the two points where it meets the double cubic, and hence these count for four intersections,

$$[C^3_{1,1}\, C^2_{3,1}] = 3 + 6 - 5 = 4;$$

also

$$[C^3_{1,1}\, C_{b,\beta}] = 3\beta + b - 5\beta = b - 2\beta,$$

and

$$[C^2_{3,1}\, C_{b,\beta}] = 6\beta + b - 5\beta = b + \beta,$$

so that any curve $C_{b,\beta}$ has $2b-\beta$ points on $C^2_{3,1}$ and $C^3_{1,1}$ together, as we have already seen. Since the formula holds for the frame, it holds for $C_{3,1}$. In general, since there are two points of $C^2_{3,1}$ on the triple line, and $2b-\beta$ points of $C_{b,\beta}$ on the triple line and $C^2_{3,1}$ together, we can make $S_\nu$ contain the triple line, $C^2_{3,1}$ and $C_{b,\beta}$ if we place $\nu+1$ points of $S_\nu$ on the triple line, $3\nu-1$ additional points on $C^2_{3,1}$ and $b\nu+1-(2b-\beta)$ additional points on $C_{b,\beta}$. If we choose $\nu=\dfrac{2b+8+n}{5}$, where $n=0$, 1, 2, 3, or 4, according as $2b+8$ is congruent to 0, 4, 3, 2, or 1 (mod. 5) respectively, and make $S_\nu$ cut out $n$ generators in each case, we will have for the order of the residual,

$$r=2b+8+n-(b+9+n)=b-1.$$

To do this we must have

$$\frac{5}{2}\left(\frac{2b+8+n}{5}\right)\left(\frac{2b+3+n}{5}\right)+4-(b+4)\left(\frac{2b+8+n}{5}\right)$$
$$+2b-\beta-1-n\left(\frac{2b+8+n}{5}-1-\beta\right)\geqq 0,$$

*i. e.*,
$$2(5-n)b-(n+3)n+10(n-1)\beta-10\geqq 0.$$

We have the following five cases:

1st Case: $n=0$;

$$\nu=\frac{2b+8}{5};\ b=6,\ 11,\ 16,\ \ldots\ \left.\right\}\ 10b-10\beta-10\geqq 0,\ \beta\leqq\frac{b}{2},\ b\geqq 2.$$

2nd Case: $n=1$;

$$\nu=\frac{2b+9}{5};\ b=3,\ 8,\ 13,\ \ldots\ \left.\right\}\ 8b-14\geqq 0,\qquad b\geqq 2.$$

3rd Case: $n+2$;

$$\nu=\frac{2b+10}{5};\ b=5,\ 10,\ 15,\ \ldots\ \left.\right\}\ 6b+10\beta-20\geqq 0,\ \beta\geqq 1,\ b\geqq 2.$$

4th Case: $n=3$;

$$\nu=\frac{2b+11}{5};\ b=7,\ 12,\ 17,\ \ldots\ \left.\right\}\ 4b+20\beta-28\geqq 0,\ \beta\geqq 1,\ b\geqq 2.$$

5th Case: $n=4$;

$$\nu=\frac{2b+12}{5};\ b=4,\ 9,\ 14,\ \ldots\ \left.\right\}\ 2b+30\beta-38\geqq 0,\ \beta\geqq 1,\ b\geqq 4.$$

The condition is therefore satisfied for all values of $b$, and since the formula holds for $b=3$ it holds for all curves on the scroll.

## IV. *Quintic Scroll with a Triple Linear Director, a Double Conic and a Double Generator.* $F(C^3_{1,1}, C^2_{2,1})$.

This scroll is somewhat like the previous one where the double cubic has been replaced by a double conic and a double generator. The double conic has one point on the triple director and one point on the double generator, the double generator meeting both conic and triple director, for a plane through the triple director and double generator cuts the double conic twice, once on the triple director and once on the double generator because the double conic cannot lie in the same plane with either. The triple director is a $C^3_{1,1}$, the double conic a $C^2_{2,1}$ and the double generator a $C^2_{1,0}$, since no generator meets the double generator.

1. *Proof of Theorem* I. This is similar to the proof for Scroll III. If five points of $S_2$ be chosen on the double conic, $C^2_{2,1}$, two additional points on $C^3_{1,1}$ and one additional point on the double generator, making eight points in all, the $S_2$ will contain all these and cut out one generator $(10-4-3-2=1)$ which can be chosen at will by placing on it the ninth point for the determination of $S_2$, since each generator already meets $S_2$ once on $C^3_{1,1}$ and once on $C^2_{2,1}$. Therefore any curve $C_b$ meets $C^3_{1,1}$, $C^2_{2,1}$ and $C^2_{1,0}$ in a fixed number of points, say $2b-\beta$, and every generator in the same number of points, $\beta$. For the number of intersections of $C_{b,\beta}$ with each of these the formula gives:

$$\begin{aligned}
[C^3_{1,1}\ C_{b,\beta}] &= 3\beta+b-5\beta = b-2\beta\\
[C^2_{2,1}\ C_{b,\beta}] &= 4\beta+b-5\beta = b-\ \beta\\
[C^2_{1,0}\ C_{b,\beta}] &= 2\beta+0-0 \ = \quad 2\beta\\
\hline
\text{Total}\ldots\ldots\ldots\ldots\ldots &= 2b-\beta
\end{aligned}$$

Also

$$[C^2_{2,1}\ C^3_{1,1}] = 4+3-5=2,\quad [C^2_{1,0}\ C^2_{2,1}] = 2 \text{ and } [C^2_{1,0}\ C^3_{1,1}] = 2,$$

which are correct results, since, at the points where these intersect, each lies on each of the two sheets through the point and there are therefore two intersections at each point.

2. *Plane Curves.* An arbitrary plane will cut out a plane quintic having three double points, one on the double generator and two on the double conic, and one triple point; if the plane of the quintic passes through the intersection of the double conic and triple director, a double point unites with a triple point, forming a tacnode with a branch through it, and if the plane passes through the

point of intersection of the double conic and double generator, two double points unite to form a tacnode. In any case, the quintic intersects the double conic four times, twice at each double point or tacnode, for, in case the tacnode has a branch through it, one of the branches of the quintic lies on a sheet that does not go through the double conic; $[C_{5,1}\ C^2_{2,1}] = 5 + 4 - 5 = 4$. The plane of the double conic cuts out one generator that meets the conic once where the conic meets the triple director and once elsewhere when considered as lying in the plane; but considered as lying on the scroll, the generator, at the point on the triple director, lies on the sheet that does not pass through the conic, and hence there is only one intersection of the generator and the conic which is at the other point in the plane.

In general, a plane through one generator cuts out a plane quartic, having three double points, one on the triple director where the generator meets it, one on the conic where the generator does not meet it and one on the double generator. A plane through two of the three generators that meet in a point on the triple director cuts out a plane cubic with a double point on the double generator. No conic can exist except the double conic and there is therefore no triple tangent plane.

[It seems probable that a conic could exist if there were a cuspidal generator instead of a double generator, which conic would lie in the tangent plane of the cuspidal edge.]

3. *Twisted Curves.* Since two generators lie in a plane, every cubic is a $C_{3,1}$, meeting the triple director once, the double conic twice and the double generator twice. If we pass a quadric surface, $S_2$, through the $C_{3,1}$, the double generator and the triple director by choosing seven points of $S_2$ on $C_{3,1}$, one more on the double generator, and one additional point on the triple director, the order of the residual will be $10 - 3 - 2 - 3 = 2$ and the residual must consist of two generators, since every generator already meets $S_2$ twice. The formula holds therefore for $C_{3,1}$.

We have seen that the triple director and double conic have one point in common and that the double generator meets both, and that altogether they meet $C_{b,\beta}$ in $2b - \beta$ points; we can, therefore, make $S_\nu$ contain the triple director, double generator, double conic, and $C_{b,\beta}$ by choosing $\nu + 1$ points of $S_\nu$ on the triple director, $\nu$ additional points on the double generator, $2\nu - 1$ additional points on the double conic and $b\nu + 1 - (2b - \beta)$ additional points on $C_{b,\beta}$. The number of arbitrary points remaining for the determination of $S_\nu$ will then be

$$\tfrac{5}{2}\nu(\nu-1)+4-[(b+4)\nu-2a+\alpha+1]$$
$$=\tfrac{5}{2}\nu(\nu-1)-(b+4)\nu+2b-\alpha+3,$$

which is the same as for Scroll III, and by taking $\nu=\dfrac{2b+8+n}{5}$ as before, where $n=0$, 1, 2, 3, or 4 according as $2b+8$ is congruent to 0, 4, 3, 2, 1 (mod. 5), respectively, and making $S_\nu$ cut out $n$ generators in each case, we will have

$$r=2b+8+n-(b+9+n)=b-1$$

as before, and we can prove, in the same manner as for Scroll III, that the formula holds for all curves on the scroll.

## V. *Quintic Scroll with a Triple Director, a Double Director and two Double Generators.* $F(C^3_{1,1}\ C^2_{1,1})$.

The four lines form a *gauch quadrilateral* in which the two directors do not intersect and the two double generators do not intersect;

$$[C^3_{1,1}\ C^2_{1,1}]=3+2-5=0 \text{ and } [C^2_{1,0}\ C^2_{1,0}]=0;$$

also $[C^3_{1,1}\ C^2_{1,0}]=0+2-0=2$, which is correct, since the double generator lies on two sheets through the triple director and has two intersections with the triple director at the point where they meet; the same is true for the double director and a double generator, $[C^2_{1,1}\ C^2_{1,0}]=0+2-0=2$.

1. *Proof of Theorem* I. Pass a quadric surface, $S_2$, through the four vertices and through one additional point in each side of the gauch quadrilateral; it will cut out all four lines, making the order of the residual

$$10-3-2-2-2=1,$$

*i. e.*, the residual is a generator that may be chosen at will by placing on it the ninth point for the determination of $S_2$. Therefore, as before, every generator meets $C_b$ in the same number of points, say $\beta$, and $C_{b,\beta}$ has $2b-\beta$ points on the four fixed lines taken together. The formula gives $2\beta$ points of intersection of $C_{b,\beta}$ and each double generator, $b-2\beta$ points for $C_{b,\beta}$ and the triple director, and $b-3\beta$ points for the double director and $C_{b,\beta}$, making in all $2b-\beta$, as before.

2. *Plane Curves.* Every plane quintic has one triple point and three double points unless its plane goes through a vertex of the gauch quadrilateral,

each of the two vertices on the triple director giving rise to a tacnode with a branch through it and each of the two vertices on the double director giving rise to a tacnode. A plane through a single generator cuts out a plane quartic that has three double points, one on the triple director, and one on each double generator, and meets the double director where the generator meets it. Since every plane through two distinct generators that meet in a point on the double director cuts out the triple director, and every plane through two generators that meet in a point on the triple director cuts out the double director and therefore the third generator through the point on the triple director, the plane of every cubic must pass through one of the double generators; conversely, every plane through either double generator that does not contain a director cuts out a plane cubic that has a double point where the plane cuts the other double generator, passes once through the point on the triple line and does not meet the double director at all, and we have

$$[C^3_{1,1}\ C_{3,1}] = 3 + 3 - 5 = 1, \quad [C^2_{1,1}\ C_{3,1}] = 2 + 3 - 5 = 0.$$

There are then two sets of plane cubics, each set lying in a sheaf of planes whose axis is a double generator. It may be observed that there exists an infinite number of triple tangent planes, since the three generators through the same point on the triple line lie in a plane.

3. *Twisted Curves.* Since three generators lie in a plane, $\beta \overset{=}{<} \frac{b}{3}$, and every cubic is a $C_{3,1}$ that has one point on the triple director, two points on each double generator and does not meet the double director at all. If we pass a quadric surface, $S_2$, through $C_{3,1}$ by placing seven points of $S_2$ on it and choose the remaining two points of $S_2$ on the triple director, then $S_2$ will cut out $C_{3,1}$, the triple director and the two double generators $(3 + 3 + 2 + 2 = 10)$ which form the complete intersection. The formula holds therefore for the $C_{3,1}$. By making $S_\nu$ pass through the vertices of the gauch quadrilateral and $\nu - 1$ additional points on each of the four sides, and through $b\nu + 1 - (2b - \beta)$ additional points of $C_{b,\beta}$, we will make $S_\nu$ cut out $C_{b,\beta}$, the triple and double directors, and the two double generators, and the remaining number of arbitrary points of $S_\nu$ will be

$$\tfrac{5}{2}\nu(\nu - 1) + 4 - [(b + 4)\nu - 2b + \beta + 1]$$
$$= \tfrac{5}{2}\nu(\nu - 1) - (b + 4)\nu + 2b - \beta + 3.$$

By choosing $\nu = \dfrac{2b+8+n}{5}$, where $n = 0, 1, 2, 3$ or $4$, according as $2a + 8$ is congruent to 0, 4, 3, 2 or 1 (mod. 5), respectively, and making $S_\nu$ cut out $n$ generators in each case, the residual will be of order

$$r = 2b + 8 + n - (b + 9 + n) = b - 1,$$

and it can be shown in the same manner as for Scroll III, that the formula holds for every curve on the scroll.

## VI. *Quintic Scroll with a Double Director and a Double Twisted Quintic with a Triple Point.* $F(C^2_{1,1}, C^2_{5,2})$.

A plane through the double director cuts out three generators that intersect, two and two, in three non-collinear points on the double quintic (the three points forming a triangle, two on each generator). The quintic is therefore a $C^2_{5,2}$ and meets the double director, $C^2_{1,1}$, at two points giving four intersections, $[C^2_{5,2}\ C^2_{1,1}] = 10 + 4 - 10 = 4$, each point counting for two intersections, since both line and curve lie on both sheets.

1. *Proof of Theorem* I. By choosing the triple point and 13 other points of $C^2_{5,2}$ as points of $S_3$ and two additional points of $S_3$ on the double director, we can make $S_3$ cut out $C^2_{5,2}$ and $C^2_{1,1}$. Each generator will then meet $S_3$ twice on $C^2_{5,2}$ and once on $C^2_{1,1}$, and consequently the residual must consist entirely of three generators which can be chosen at will since there are still three arbitrary points remaining for the determination of $S_3$. Any two of these generators may be kept fixed and the third one varied so as to include, in succession, all the other generators of the scroll, and since a fixed number of points of a curve, $C_b$, lie on $C^2_{1,1}$ and $C^2_{5,2}$ together, it is evident that the same number of points, say $\beta$, lie on each generator and that $3b - 3\beta$ points of $C_{b,\beta}$ lie on $C^2_{1,1}$ and $C^2_{5,2}$ together;

$$[C^2_{1,1}\ C_{b,\beta}] = 2\beta + b - 5\beta = b - 3\beta$$

and

$$[C^2_{5,2}\ C_{b,\beta}] = 10\beta + 2b - 10\beta = 2b$$

making a total of $3b - 3\beta$.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points; if the plane of the proper quintic goes through either of the two points common to the double director and double quintic, two of the double points unite to form a tacnode. A plane through only one generator cuts out a plane quartic having three double points where the plane cuts $C^2_{5,2}$, not on the

generator. A plane through two generators that pass through the same point on the double director cuts out a plane cubic with a double point where the plane cuts $C^2_{5,2}$ not on either generator. No conic exists.

3. *Twisted Curves.* Since three generators lie in a plane, every cubic is a $C_{3,1}$ and does not meet the double director, $[C_{3,1}\, C^2_{1,1}] = 3 + 2 - 5 = 0$; it must therefore meet $C^2_{5^2,2}$ in $3(b - \beta) = 6$ points, $[C_{3,1}\, C^2_{5,1}] = 6 + 10 - 10 = 6$. If we pass $S_4$ through 21 points of $C^2_{5,2}$, three additional points of the double director and seven additional points of $C_{3,1}$, $S_4$ will cut out the two curves and the double director, and the residual will consist entirely of generators, since each generator already meets $S_4$ twice on $C^2_{5,2}$, once on $C_{3,1}$ and once on the double director, and can not meet it again without lying entirely on it. Therefore the formula holds for every $C_{3,1}$. If, in general, we make $S_\nu$ pass through $5\nu - 1$ points of $C^2_{5,2}$ including the triple point, through $\nu - 1$ additional points of the double director and $b\nu + 1 - 3(b - \beta)$ additional points of $C_{b,\beta}$, then $S_4$ will cut out the frame and $C_{b,\beta}$, and the number of arbitrary points remaining for the determination of $S_4$ will be

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{(b + 6)\nu - 1 - 3(b - \beta)\}.$$

By choosing $\nu = \dfrac{2b + 11 + n}{5}$, where $n = 1, 2, 3, 4$ or $5$, according as $2b + 11$ is congruent to 4, 3, 2, 1, or 0 (mod. 5) respectively, and making $S_\nu$ cut out $n$ generators in each case, the residual will be of order $b - 1$ and it can be shown in the same manner as for Scroll II that the formula holds for all curves on the scroll.

## VII. *Quintic Scroll with a Double Director, a Double Generator, and a Double Twisted Quartic having a Double Point.* $F(C^2_{1,1}\, C^2_{4,2})$.

A plane through the double director cuts out three generators whose mutual intersections form a triangle with the vertices on the double quartic, two on each generator; as the plane is revolved around the double director it comes to a position in which it passed through the double point of the quartic, and in this position the plane cuts out the double generator and a single generator, two of the generators uniting as two of the vertices of the triangle fall together at the double point of the quartic; the double generator therefore passes through the double point of the quartic and one other point, while the single generator in this plane passes through the double point of the quartic, on a sheet different from those through the double generator; thus the double generator has two intersections with the quartic at the double point and two more at the other point,

$[C^2_{1,0}\ C^2_{4,2}] = 4$, while the single generator has two intersections with the quartic and none with the double generator at the double point of the quartic, $[C_{1,0}\ C^2_{4,2}] = 2$ and $[C_{1,0}\ C^2_{1,0}] = 0$. The double director and double quartic have one point in common.

1. *Proof of Theorem* I. By passing $S_3$ through the frame and the double generator, *i. e.*, by choosing the double point and 11 other points of $C^2_{4,2}$, three additional points of the double director, and one additional point of the double generator as points of $S_3$, Theorem I can be proved in exactly the same way as for Scroll VI.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points, four on $C^2_{4,2}$, one on the double director and one on the double generator. A plane through one generator cuts out a plane quartic with three double points, one on the double generator and two on $C^2_{4,2}$. A plane through two generators that meet on the double director cuts out a plane cubic with a double point on the double generator, while a plane through the double generator cuts out a plane cubic with a double point on $C^2_{4,2}$, unless the plane also cuts out the double director.

3. *Twisted Curves.* Since the double quartic, $C^2_{4,2}$, and the double generator here, take the place of the double quintic on Scroll VI—counting for 10 in the order of the section by $S_\nu$ and requiring the same number of points of $S_\nu$, *i. e.*, $4\nu$ for $C^2_{4,2}$ (including the double point) and $\nu - 1$ additional points for $C^2_{1,0}$, or $5\nu - 1$ in all—the proof of the formula for this scroll is exactly the same as for Scroll VI.

## VIII. *Quintic Scroll with a Double Conic and a Double Twisted Quartic having a Double Point.* $F(C^2_{2,1}\ C^2_{4,2})$.

A plane through two generators that meet on the double conic cuts out a cubic; there are six double points in the section, two on the double conic and four on the double quartic; the generators in this plane can not meet the conic again, so that the cubic has a double point on the double conic; the four double points of the section that are on the quartic lie two and two on each generator, through each of which the cubic passes. The double conic is therefore a $C^2_{2,1}$ and the double quartic is a $C^2_{4,2}$.

Schwarz has shown that the double conic goes through the double point of the quartic and meets the quartic in two other points; at each of these three points the conic and quartic intersect twice, since each lies on both sheets, making six intersections and $[C^2_{2,1}\ C^2_{4,2}] = 8 + 8 - 10 = 6$.

1. *Proof of Theorem* I. In the same way as for Scroll II, we can pass $S_3$ through the frame, obtaining three generators as the residual, and prove Theorem I in exactly the same way as for Scroll II.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points, two on $C^2_{2,1}$ and four on $C^2_{4,2}$; if the plane goes through the double point of $C^2_{4,2}$ and does not contain a generator, the section is a plane quintic with a triple point formed by the union of two of the double points on $C^2_{4,2}$ with one double point on $C^2_{2,1}$. The plane of the double conic cuts out one generator, as in the case of Scroll IV, which meets the conic once at the double point of the twisted quartic $C^2_{4,2}$, and once elsewhere when considered as lying in the plane, but has only one intersection with the conic when considered as lying on the scroll, since at the double point of $C^2_{4,2}$ the generator lies on a sheet that does not pass through the conic there. Any plane, except the plane of the double cone that passes through one generator, cuts out a plane quartic having two double points on $C^2_{4,2}$ and one on $C^2_{2,1}$. A plane through two generators that meet on $C^2_{2,1}$ cuts out a plane cubic with a double point on $C^2_{2,1}$, and a plane through two generators that meet on $C^2_{4,2}$ cuts out a cubic with a double point on $C^2_{4,2}$.

3. *Twisted Curves.* Since the double quartic and double conic take the place of the double sextic on Scroll II, counting 12 in the order of the section by $S_\nu$ and requiring the same number of points of $S_\nu$—viz., $4\nu+1$ for $C^2_{4,2}$, if the double point is not chosen, and $2\nu-2$ additional points for $C^2_{2,1}$, or $6\nu-1$ in all—the proof of the formula for this scroll is the same as for Scroll II.

### IX. *Quintic Scroll with Three Double Conics.* $F(C^2_{2,1}\, C^2_{2,1}\, C^2_{2,1})$.

The three double conics form a degenerate double sextic curve; each conic meets each of the other two in two points, and they have one point in common which takes the place of the triple point of the sextic, or, as Schwarz has pointed out, the planes of the three conics are three sides of a tetrahedron whose vertex is the point common to the three conics and whose base is the plane through the three other points of intersection of the conics, two and two. Since an arbitrary generator can meet the conic once only, each double conic is a $C^2_{2,1}$, and has three intersections with each of the other two, two at a point not common to all, where each conic lies on both sheets, and one at the common point where each conic meets the generator in the plane of the other on one sheet and the conic once on the other sheet; $[C^2_{2,1}\, C^2_{2,1}] = 4 + 4 - 5 = 3$.

Therefore the formula holds for the curves of the frame. The proof of

Theorem I is the same as for Scroll II, since $S_3$ can be passed through the frame and three arbitrary generators, and the proof of the formula is the same as for Scroll II, since the frame counts for 12 in the section by $S_\nu$ and requires only $6\nu - 2$ points of $S_\nu$. $[C^2_{2,1}\, C_{b,\beta}] = 4\beta + b - 5\beta = b - \beta$, so that any curve, $C_{b,\beta}$, meets the three conics together in $3b - 3\beta$ points, which is the same number as that for the proper sextic on Scroll II.

### X. *Quintic Scroll with a Double Generator and a Double Twisted Quintic having a Double Point.* $F(C^2_{5,3})$.

The double quintic and double generator together form a degenerate sextic with a triple point formed by the double generator passing through the double point of the quintic, but not in the plane of the tangents to the curve. This scroll differs from Scroll VI by having a double generator instead of a double director, in consequence of which each generator on this scroll meets the double quintic three times and it is therefore a $C^2_{5,3}$. The double generator has six intersections with the double quintic; $[C^2_{1,0}\, C^2_{5,3}] = 6$. The proof of Theorem I is the same as for Scroll II, since $S_3$ can be made to cut out $C^2_{5,3}$, the double generator and three arbitrary generators, and the proof of the formula is the same as for Scroll II, since the double quintic and double generator count for 12 in the section by $S_\nu$ and require $6\nu - 1$ points of $S_\nu$.

## (B) SCROLLS OF DEFICIENCY ONE. ($\rho = 1$.)

### XI. *Quintic Scroll with Double Twisted Quintic That is Met Three Times by Each Generator.* $F(C^2_{5,3})$.

This scroll differs from Scroll X by having no double point on the quintic and consequently no double generator, which makes every plane curve of *deficiency one.*

1. *Proof of Theorem* I. If 16 points of $S_3$ be taken on $C^2_{5,3}$, $S_3$ will contain this curve and will cut out of the scroll, in addition, five generators, since each generator already meets $S_3$ in three points on $C^2_{5,3}$. There remain three points of $S_3$ that can be arbitrarily chosen so that three of the five generators can be chosen at will. Therefore by a method employed for Scroll II it can be shown that only one infinite set of generators exists, and each generator meets $C_b$ in the same number of points, $\beta$. There must then be $3b - 5\beta$ points of $C_{b,\beta}$ on $C^2_{5,3}$, and the formula gives

$$[C^2_{5,3}\, C_{b,\beta}] = 10\beta + 3b - 15\beta = 3b - 5\beta,$$

so that the formula holds for the frame.

2. *Plane Curves.* Every plane not containing a generator cuts out a plane quintic having five double points, and no other kind of a plane quintic can exist A plane through one generator cuts out a plane quartic having two double points, and a plane through two generators cuts out a plane cubic without a double point.

3. *Twisted Curves.* Since two generators lie in a plane, $\beta \leqq \frac{b}{2}$, and therefore every cubic is a $C_{3,1}$. If 21 points of $S_4$ be chosen on $C^2_{5,3}$ and nine additional points on $C_{3,1}$, $S_4$ will cut out these two curves from the scroll and the residual intersection will consist entirely of generators, since each generator already meets $S_4$ three times on $C^2_{5,3}$ and once on $C_{3,1}$. Therefore the formula holds for $C_{3,1}$. In general $S_\nu$ may be passed through $C^2_{5,3}$ and any curve $C_{b,\beta}$ by choosing $5\nu + 1$ points of $S_\nu$ on $C^2_{5,3}$ and $b\nu + 1 - (3b - 5\beta)$ additional points of $S_\nu$ on $C_{h,\beta}$, which will leave

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{(b + 5)\nu - 3b + 5\beta + 2\}$$
$$= \tfrac{5}{2}\nu(\nu - 1) - (b + 5)\nu + 3b - 5\beta + 2$$

arbitrary points for the determination of $S_\nu$. Choosing $\nu = \frac{2b + 13 + n}{5}$, where $n = 0, 1, 2, 3$ or $4$ according as $2b + 13$ is congruent to 0, 4, 3, 2 or 1 (mod. 5), respectively, and making $S_\nu$ cut out $n + 4$ generators in each case, we have the condition,

$$\frac{5}{2}\left(\frac{2b + 13 + n}{5}\right)\left(\frac{2b + 8 + n}{5}\right) - (b + 5)\left(\frac{2b + 13 + n}{5}\right)$$
$$+ 3b - 5\beta + 2 - (n + 4)\left(\frac{2b + 13 + n}{5} + 1 - 3 - \beta\right) \geqq 0,$$

*i. e.*,
$$2(5 - n)b + (n - 1)10\beta - n(n + 3) - 30 \geqq 0,$$

where
$$1 \leqq \beta \leqq \frac{b}{2}.$$

If this condition is satisfied, the order of the residual is

$$r = 2b + 13 + n - \{10 + b + (n + 4)\} = b - 1.$$

There are five cases to consider, viz.:

1st Case: $b = 6,\ 11,\ \ldots\ldots;$

$$\left.\begin{array}{l} 2b + 13 \equiv 0 \text{ (mod. 5)};\ n = 0,\ \nu = \dfrac{2b + 13}{5} \end{array}\right\} \begin{array}{c} \text{Condition is} \\ 10b - 10\beta - 30 \geqq 0 \end{array} \left\{\begin{array}{l} \beta = \dfrac{b}{2},\ b \geqq 6 \\ \beta = 1\ \ ,\ b \geqq 4 \end{array}\right\}.$$

Satisfied for $b = 6,\ 11$, etc.

2nd Case: $b = 3, 8, 13, \ldots;$

$$2b + 13 \equiv 4 \ (\text{mod. } 5);\ n = 1,\ \nu = \frac{2b+14}{5} \left\} \begin{array}{l} \text{Condition is} \\ 8a - 34 \geqq 0. \end{array} \right.$$

Satisfied for $b \geqq 5$, *i. e.*, for $b = 8, 13$, etc.

3rd Case: $b = 5, 10, 15, \ldots;$

$$2b + 13 \equiv 3 \ (\text{mod. } 5);\ n = 2,\ \nu = \frac{2b+15}{5} \left\} \begin{array}{c} \text{Condition is} \\ 6b + 10\beta - 40 \geqq 0. \end{array} \right\} \left\{ \begin{array}{c} \text{Satisfied if} \\ \beta = \frac{b}{2},\ b \geqq 4 \\ \beta = 1\ ,\ b \geqq 5 \end{array} \right\}$$

and $\therefore$ for $b = 5, 10, 15$, etc.

4th Case: $b = 7, 12, 17, \ldots;$

$$2b + 13 \equiv 2 \ (\text{mod. } 5);\ n = 3,\ \nu = \frac{2b+16}{5} \left\} \begin{array}{c} \text{Condition is} \\ 4b + 20\beta - 48 \geqq 0. \end{array} \right\} \left\{ \begin{array}{c} \text{Satisfied if} \\ \beta = \frac{b}{2},\ b \geqq 4 \\ \beta = 1\ ,\ b \geqq 7 \end{array} \right\}$$

and $\therefore$ for $b = 7, 12, 17$, etc.

5th Case: $b = 4, 9, 14, 19, \ldots.$

$$2b + 13 \equiv 1 \ (\text{mod. } 5);\ n = 4,\ \nu = \frac{2b+17}{5} \left\} \begin{array}{c} \text{Condition is} \\ 2b + 30\beta - 58 \geqq 0. \end{array} \right\} \left\{ \begin{array}{c} \text{Satisfied if} \\ \beta = \frac{b}{2},\ b \geqq 4 \\ \beta = 1\ ,\ b \geqq 14 \\ \beta \geqq 2\ ,\ b \geqq 0 \end{array} \right\}$$

and $\therefore$ for $b = 14, 19, 24$, etc.

The only curves not included in this list which satisfy the condition are $C_{3,1}$ under case 2, for which the proof has already been given, and $C_{4,1}$, $C_{9,1}$, under case 5. Now $C_{4,1}$ can be cut out by an $S_4$ that passes through $C^2_{5,3}$ (and the residual will consist entirely of generators), for it only requires 21 points of $S_4$ on $C^2_{5,3}$ and 10 additional points on $C_{4,1}$ and we have 34 points available. $C_{9,1}$ can be cut out by an $S_5$ that passes through $C^2_{5,3}$, for it only requires 26 points of $S_5$ on $C^2_{5,3}$ and 24 additional points on $C_{9,1}$ and we have 54 points available; the residual is of order 6, for which the formula holds. Therefore the formula holds for all curves on the scroll.

## XII. *Quintic Scroll with a Double Director and a Double Conic.* $F(C^3_{1,1}\, C^2_{2,1})$.

A plane through the triple director cuts out two generators that meet on the double conic, and since the generators can not meet the double conic again, the double conic meets the triple director in one point, which counts for two intersections since the two sheets through the double conic also pass through the triple director; $[C^3_{1,1}, C^2_{2,1}] = 3 + 4 - 5 = 2$, and therefore the formula holds for the frame.

1. *Proof of Theorem* I. By choosing five points of $S_2$ on $C^2_{2,1}$ and two additional points of $S_2$ on $C^3_{1,1}$, $S_2$ will cut out the frame and three generators, two of which may be chosen at will as there are still two arbitrary points for the determination of $S_2$, and each generator already meets $S_2$ twice on the frame. Therefore, by the method employed for Scroll II, Theorem I is proven for this scroll. Every curve, $C_{b,\beta}$, on the scroll meets $S_2$ in $2b$ points, of which $3\beta$ lie on the three generators cut out by $S_2$, and consequently $2b - 3\beta$ points lie on the frame;

$$[C_{b,\beta}, C^3_{1,1}] = b + 3\beta - 5\beta = b - 2\beta,$$

and

$$[C_{b,\beta}, C^2_{2,1}] = b + 4\beta - 5\beta = b - \beta,$$

making in all $2b - 3\beta$.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with a triple point and two double points; a plane through one generator (except the plane of the double conic) cuts out a plane quartic with two double points, one on $C^3_{1,1}$ and one on $C^2_{2,1}$; a plane through two of the three generators that meet in a point on the triple director cuts out a plane cubic without a double point, the three points where a generator in its plane meets it being the point of tangency, one point on $C^3_{1,1}$ and one point on $C^2_{2,1}$. No conic except the double conic can exist, and the plane of the double conic cuts out one generator in the same manner as the plane of the double conic on Scroll IV.

3. *Twisted Curves.* Since the triple director and double conic have one point in common, and any curve, $C_{b,\beta}$, has $2b - 3\beta$ points on the frame, $S_\nu$ can be made to cut out the frame and $C_{b,\beta}$ if

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{2\nu + 1 + \nu + b\nu + 1 - (2b - 3\beta)\} \geqq 0.$$

If we choose $\nu = \dfrac{2b + 6 + n}{5}$, where $n = 0$ or $5, 1, 2, 3, 4$, according as $2b + 6$ is congruent to $0, 4, 3, 2, 1 \pmod{5}$ respectively, and make $S_\nu$ cut out $n$ generators in each case, we have the following condition:

$$\frac{5}{2}\left(\frac{2b + 6 + n}{5}\right)\left(\frac{2b + 1 + n}{5}\right) - (b + 3)\left(\frac{2b + 6 + n}{5}\right) + 2b - 3\beta + 2 - n\left(\frac{2b + 6 + n}{5} - \beta - 1\right) \geqq 0,$$

*i. e.,*

$$2(5 - n)b + 10(n - 3)\beta - n(n + 1) - 10 \geqq 0,$$

and

$$r = 2b + 6 + n - 6 - 7 - n = b - 1.$$

This condition is satisfied in the five cases as follows:

1st Case: $b = 7, 12, 17, \ldots$; $2b + 6 \equiv 0 \pmod{5}$.

If $n = 0$, $\nu = \frac{2b+6}{5}$ and $10b - 30\beta - 10 \geqq 0$, which is satisfied for $b \geqq 4$ if $\beta = 1$.

If $n = 5$, $\nu = \frac{2b+11}{5}$ and $20\beta - 40 \geqq 0$, which is satisfied for $b \geqq 0$ if $\beta \geqq 2$.

In this case, therefore, for $\beta = 1$, $\nu = \frac{2b+6}{5}$, and for $\beta > 1$, $\nu = \frac{2b+11}{5}$.

2nd Case: $b = 4, 9, 14, \ldots$; $2b + 6 \equiv 4 \pmod{5}$.

$n = 1$, $\nu = \frac{2b+7}{5}$, $8b - 20\beta - 12 \geqq 0$, which is satisfied if $\beta \leqq \frac{2b-3}{5}$.

When $\beta \geqq \frac{2b-3}{5}$, take $\nu = \beta + 2$, in which case the residual will consist entirely of generators (since each generator meets the frame twice), and the condition becomes

$$\tfrac{5}{2}(\beta + 2)(\beta + 1) - (b + 3)(\beta + 2) + 2b - 3\beta + 2 \geqq 0.$$

*i. e.*,

$$5\beta^2 + 3\beta - 2b\beta + 2 \geqq 0, \qquad \text{(Eq. 1.)}$$

which for $\beta \geqq \frac{2b-3}{5}$ gives $10 \geqq 0$, and hence the condition is satisfied.

3rd Case: $b = 6, 11, 16, \ldots$; $2b + 6 \equiv 3 \pmod{5}$.

$n = 2$, $\nu = \frac{2b+8}{5}$, $6b - 10\beta - 16 \geqq 0$, which is satisfied for $\beta \leqq \frac{3b-8}{5}$.

When $\beta \geqq \frac{2b-8}{5}$, take $\nu = \beta + 2$, and the condition above (Eq. 1) becomes $3b^2 - 23b + 50 \geqq 0$, which is satisfied for $b \geqq 0$.

4th Case: $b = 3, 8, 13, \ldots$; $2b + 6 \equiv 2 \pmod{5}$.

$n = 3$, $\nu = \frac{2b+9}{5}$, $4b - 22 \geqq 0$; satisfied for $b \geqq 6$.

For $b = 3$, take $\nu = \beta + 2 = 3$, since $\beta \leqq \frac{b}{2}$ must be 1. Eq. 1, above, becomes $4 \geqq 0$, which is satisfied.

5th Case: $b = 5, 10, 15, \ldots$; $2b + 6 \equiv 1 \pmod{5}$.

$n = 4$, $\nu = \frac{2b+10}{5}$, $2b + 10\beta - 30 \geqq 0$, satisfied if $\left\{\begin{matrix} \beta = 1, & b \geqq 10 \\ \beta \geqq 2, & b \geqq 5 \end{matrix}\right\}$.

For $b = 5$, $\beta = 1$, take $\nu = \beta + 2 = 3$, and Eq. 1 becomes

$$5 + 3 - 10 + 2 \overset{=}{>} 0,$$

which is satisfied.

Therefore, for all curves on the scroll, $\nu$ may be so chosen that the residual will consist of a curve of order less than $b$, or else entirely of generators (*e. g.* when $\nu = \beta + 2$, which applies to the case $b = 3$). The formula holds therefore for all curves on this scroll.

## XIII. *Quintic Scroll with a Triple Director, a Double Director and a Double Generator.* $F(C^3_{1,1}\ C^2_{1,1})$.

This scroll differs from Scroll V by having only one double generator instead of two; the number of intersections of this double generator with each director is the same as for Scroll V, $[C^3_{1,1}\ C^2_{1,0}] = 2$ and $[C^2_{1,1}\ C^2_{1,0}] = 2$.

1. *Proof of Theorem* I. If a quadric surface be passed through $C^3_{1,1}$ and $C^2_{1,1}$ and the double generator by placing three points of the quadric on each director and one on the double generator, the residual intersection will consist of three generators, two of which may be chosen at will by placing on them (one point on each) the two remaining points for the determination of the quadric. Therefore, by the reasoning employed for Scroll II the theorem is proved, and there will be $2b - 3\beta$ points of $C_{b,\beta}$ on the frame and the double generator together.

2. *Plane Curves.* Every plane quintic will have one triple and two double points unless its plane goes through the point of intersection of the double generator and a director, in which case the plane quintic has a tacnode if on the double director, and a tacnode with a branch through it if the point is the intersection of the triple director and the double generator. The plane quartics have two double points, and the plane cubics have none and exist only in the planes through the double generator.

3. *Twisted Curves.* By passing $S_\nu$ through $\nu + 1$ points on the triple director, $\nu + 1$ points on the double director and $\nu - 1$ additional points on the double generator, and $b\nu - (2b - 3\beta) + 1$ additional points on $C_{b,\beta}$, we make $S_\nu$ cut out both directors, the double generator, and $C_{b,\beta}$. This can be done if

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{(b + 3)\nu - (2b - 3\beta) + 2\} \overset{=}{>} 0.$$

This condition is exactly the same as that for Scroll XII, and the proof of the formula is therefore the same as for that scroll.

## XIV. *Quintic Scroll with a Double Director and a Double Twisted Quartic.* $F(C^2_{1,1}\ C^2_{4,2})$.

This scroll is similar to Scroll VII, but here no double point exists on the twisted quartic, and no double generator exists on this scroll. A plane through the double director cuts out three generators that form a triangle by intersecting two and two on three points of the twisted quartic, the other point of the twisted quartic in the plane lying on the director, as in Scroll VII.

1. *Proof of Theorem* I. Pass $S_3$ through the double director and twisted quartic by choosing four points of $S_3$ on $C^2_{1,1}$ and twelve additional points of $S_3$ on $C^2_{4,1}$. Since each generator meets the quartic twice and the director once, the residual intersection of $S_3$ and the scroll will consist of five generators, three of which may be chosen at will since three points of $S_3$ are still to be determined. Therefore, by a method analogous to the one employed for Scroll II, Theorem I may be proved, and therefore there are $3b - 5\beta$ points of $C_{b,\beta}$ on the double director and the double twisted quartic together.

2. *Plane Curves.* The plane quintic has five double points, the plane quartic has two double points and the plane cubic has no double point.

3. *Twisted Curves.* The double director and double twisted quartic form a degenerate twisted quintic, and the proof of the formula for this scroll is the same as that for Scroll XI, since $S_\nu$ can be made to pass through the frame and $C_{b,\beta}$ by placing $\nu+1$ points of $S_\nu$ on $C^2_{1,1}$, $4\nu$ additional points of $S_\nu$ on $C^2_{4,1}$ and $b\nu + 1 - (3b - 5\beta)$ additional points of $S_\nu$ on $C_{b,\beta}$, leaving

$$\{\tfrac{5}{2}\nu(\nu-1)+4\} - \{(b+5)\nu - 3b + 5\beta + 2\}$$
$$= \tfrac{5}{2}\nu(\nu-1) - (b+5)\nu + 3b - 5\beta + 2$$

arbitrary points for the determination of $S_\nu$ as before.

### (C) SCROLL OF DEFICIENCY TWO. ($\rho = 2$.)

## XV. *Quintic Scroll with a Triple Director and a Double Director.* $F(C^3_{1,1}\ C^2_{1,1})$.

This is the only quintic scroll of deficiency two. A plane through the triple director cuts out two generators that pass through the same point on the double director, and a plane through the double director cuts out three generators that pass through the same point on the triple director, and this scroll differs from Scroll XIII by not having a double generator.

1. *Proof of Theorem* I. By passing a quadric surface, $S_2$, through both directors, placing three points of $S_2$ on each, the residual intersection will be five

generators, three of which may be chosen at will by placing on each one point of the remaining three points for the determination of $S_2$. Therefore, by a method similar to the one employed for Scroll II, Theorem I can be proved and there are therefore $2b - 5\beta$ points of $C_{b,\beta}$ on the two directors taken together By the formula $[C_{b,\beta}\, C^3_{1,1}] = b - 2\beta$ and $[C_{b,\beta}\, C^2_{1,1}] = b - 3\beta$.

2. *Plane Curves.* Every plane quintic has one triple and one double point, and every quartic has one double point, and no other plane curves exist.

3. *Twisted Curves.* Let $S_\nu$, which cuts out the curve $C_{b,\beta}$, have the triple director as a line of multiplicity $\nu - \beta$; the residual in this case must consist entirely of generators, since each generator will meet $S_\nu$ $\beta$ times on $C_{b,\beta}$ and $\nu - \beta$ times on the director, *i. e.*, $\nu$ times in all, and can not meet $S_\nu$ again without lying on it and forming a part of the residual. It is only necessary then to show that a value for $\nu$ can be found such that $S_\nu$ will cut out the curve $C_{b,\beta}$ and will have the triple director as a line of multiplicity $\nu - \beta$ and shall not break up into the scroll itself, *i. e.*, a surface of order 5 with a triple line, and another surface of order $\nu - 5$ with this same line as a line of multiplicity $\nu - \beta - 3$.

If we let $x = 0$, $y = 0$, in homogeneous coordinates, be the line of multiplicity $k$ on a surface of order $m$, the number of terms in the equation of the surface will be the sum of all the terms of degree $m$ in $x$ and $y$ together, plus all the terms down to those of degree $k$ in $x$ and $y$, *i. e.*,

$$\begin{aligned}
&(m+1) + 2m + 3(m-1) + 4(m-2) + \ldots\ldots + (m-k+1)(k+1) \\
&= (m+1)[1+2+3+\ldots\ldots+m-k+1] \\
&\qquad\qquad - 2\left[1+3+6+\ldots\ldots+\frac{(m-k+1)(m-k)}{2}\right] \\
&= (m+1)\binom{m-k+2}{2} = 2\binom{m-k+2}{3}.
\end{aligned}$$

$$\left[\text{Using here the notation } \binom{p+q}{r} \equiv \frac{(p+q)(p+q-1)\ldots\ldots(p+q-r+1)}{1\,.\,2\,.\,3\ldots\ldots r}.\right]$$

Since a plane through the triple director cuts out two generators, there are $b - 2\beta$ points of $C_{b,\beta}$ on the triple director, which count for $(b - 2\beta)k$ points of intersection with a surface having this line of multiplicity $k$. In the present case $m = \nu$ and $k = \nu - \beta$ and

$$(\nu+1)\binom{\beta+2}{2} - 2\binom{\beta+2}{3} - 1 - b\beta - 2\beta\nu + 2\beta^2 - 1 \geqq 0.$$

Now when $\beta = 1$, it is impossible for $S_\nu$ to have a component of order $\nu - 5$ with a line of multiplicity $\nu - \beta - 3 = \nu - 4$, so that for $\beta = 1$,

$$(\nu + 1)\binom{3}{2} - 2\binom{3}{3} - 1 - b - 2\nu + 2 - 1 \geqq 0,$$

or

$$\nu \geqq b - 1.$$

For $\beta = 2$, the only way in which a surface of order $\nu - 5$ could have a line of multiplicity $\nu - \beta - 3 = \nu - 5$ is for the surface to break up into $\nu - 5$ planes through the line. If then we impose the condition that $S_\nu$ shall contain a line that is not on the scroll, $S_\nu$ can not have the scroll as a component, so that for $\beta = 2$

$$(\nu + 1)\binom{4}{2} - 2\binom{4}{3} - 2b - 4\nu + 6 - (\nu + 1) \geqq 0,$$

$$\nu \geqq 2b - 3.$$

For $\beta \geqq 3$, $S_\nu$ may break up into the scroll and $S_{\nu-5}$ with the triple director as a line of multiplicity $\nu - \beta - 3 \leqq \nu - 6$ on $S_{\nu-5}$; to prevent this we must make $S_\nu$ pass through a number of arbitrary points not on the scroll equal to one more than the number of arbitrary coefficients in the equation of an $S_{\nu-5}$ having a line of multiplicity $\nu - \beta - 3$, which is (since $m = \nu - 5$, $k = \nu - \beta - 3$, and $m - k + 2 = \beta$)

$$(\nu + 4)\binom{\beta}{2} - 2\binom{\beta}{3};$$

we have therefore

$$(\nu + 1)\binom{\beta + 2}{2} - 2\binom{\beta + 2}{3} - 1 - (\nu - 4)\binom{\beta}{2} + 2\binom{\beta}{3}$$
$$- b\beta - 2\beta\nu + 2\beta^2 - 1 \geqq 0,$$

from which

$$\nu \geqq b\beta - \frac{5\beta^2 - \beta - 2}{2}; \quad 3 \leqq \beta \leqq \frac{b}{3}.$$

Therefore a value for $\nu$ can always be found such that the residual will consist entirely of generators, and therefore the formula holds for all curves on the scroll.